1 Arithmetic Series

What's the difference between an arithmetic *sequence* and an arithmetic *series*?

When we add the terms of a sequence together, we get a series!

For example,

$3, 5, 7, 9, 11$ is a *sequence*, but

$3 + 5 + 7 + 9 + 11$ is a *series*.

We use the symbol S_n to represent the sum of the terms in an arithmetic series. Above, we see that $S_5 =$ ____________.

2 Investigate!

To examine the method we use for determining a formula for the sum of n terms of an arithmetic series, we often tell the story of *Karl Gauss* (1777-1855):

One day Gauss' teacher asked his class to add together all the numbers from 1 to 100, assuming that this task would occupy them for quite a while. He was shocked when young Gauss, after a few seconds thought, wrote down the answer 5050. The teacher couldn't understand how his pupil had calculated the sum so quickly in his head, but the eight year old Gauss pointed out that the problem was actually quite simple.

Here, we use a similar method to determine the answer.

$$S_{100} = 1 + 2 + 3 + 4 + 5 + ... + 96 + 97 + 98 + 99 + 100$$
$$S_{100} = 100 + 99 + 98 + 97 + 96 + ... + 5 + 4 + 3 + 2 + 1$$

Add the rows and complete the work to show that $S_{100} = 5050$.

$2S_{100} = 101 + 101+$

3 Formulas for the Sum of an Arithmetic Series

In an arithmetic series of n terms:

the first term is a, the second term is $a + d$, and the last term, $t_n = a + (n - 1)d$.

The sum of n terms of the series can be written as:

$S_n = a + (a + d) + (a + 2d) + ... + (a + (n - 3)d) + (a + (n - 2)d) + (a + (n - 1)d)$

or

$S_n = (a + (n - 1)d) + (a + (n - 2)d) + (a + (n - 3)d) + ... + (a + 2d) + (a + d) + a$

adding these two lines together gives us:

$2S_n = (2a + (n - 1)d) + (2a + (n - 1)d) + (2a + (n - 1)d) + ... + (2a + (n - 1)d) + (2a + (n - 1)d) + (2a + (n - 1)d)$

$2S_n = n(2a + (n - 1)d)$

Dividing this by 2 gives us the **formula for the sum of n terms of an arithmetic series:**

$$\boxed{S_n = \frac{n[2a+(n-1)d]}{2}}$$

This formula connects a, d, n, and S_n – if any three of these values are known, the fourth can be determined!

However! When the common difference of the series is not know, *another* formula for the sum of n terms of an arithmetic series can be formed by replacing $a + (n - 1)d$ with t_n to give us:

$$\boxed{S_n = \frac{n(a+t_n)}{2}}$$

We can think of this formula as being the average of the first and last term, multiplied by the number of terms.

4 Examples

1. Determine the sum of the first fourteen terms of the arithmetic series $9 + 15 + 21 + ...$

2. Determine the sum of 22 terms of an arithmetic sequence with $t_1 = 18$ and $t_{22} = 45$.

3. Find the sum of the terms in the sequence $17, 12, 7, ... - 38$.

4. Darius starts a new job, working for $16 000/year. Each year, he gets a raise of $850.
   (a) Calculate his salary in the twelfth year.

   (b) How much money did he earn in total over his twelve years at the company?

5 Investigate!

Hannah was given two questions on an assignment. The first question is listed below.

"Find the first four terms of the series defined by $S_n = 2n^2 - n$."

(a) Complete her work below to find the first four terms

$S_n = 2n^2 - n$

$S_1 = 2(1)^2 - 1 = 1$ $\quad t_1 = S_1$ $\quad t_1 = 1$

$S_2 = 2(2)^2 - 2 = 6$ $\quad S_2 = S_1 + t_2$ $\quad t_2 = S_2 - S_1$ $\quad t_2 = 6 - 1$ $\quad t_2 = 5$

$S_3 = 3(2)^2 - 3 = 15$ $\quad S_3 = S_2 + t_3$

(b) Express t_{10} in terms of S.

(c) Express t_n in terms of S.

(d) The second question Hannah had was: "Find t_n if $S_n = 2n^2 = n$."

(i) Find t_n using $t_n = a + (n-1)d$

(ii) Find t_n using the formula in c)

Remember this! $\boxed{t_n = S_n - S_{n-1}, n \geq 2, n \in \mathbb{N}}$